Some hints for Homework 3.

1. Here is the the derivation for the 2-stage RK methods. The 3-stage is the same in principle but you have to include one more order in the multivariable and single-variable Taylor expansions.

RK2

$$y^{(0)} = y^{(k)}.$$

$$K_1 = f(y^{(0)}), \quad y^{(1)} = y^{(k)} + hb_{21}K_1$$

$$K_2 = f(y^{(1)}), \quad y^{(2)} = y^{(k)} + h(b_{31}K_1 + b_{32}K_2) = y^{(k+1)}$$

$$y_i^{(k+1)} = y_i^{(k)} + hb_{31}f_i(y^{(k)}) + hb_{32}f_i(y^{(k)} + \underbrace{hb_{21}K_1}_{\text{"}v\text{"}})$$

$$K_1 = f(y^{(0)})$$

$$= y_i^{(k)} + hb_{31}f_i(y^{(k)})$$

$$+ hb_{32}\Big[f_i(y^{(k)}) + hb_{21}\big(\underbrace{\partial_1 f_i \cdot f_1 + \partial_2 f_i \cdot f_2}_{\text{eval. at } y^{(k)}}\big) + O(h^2)\Big]$$

$$= y_i^{(k)} + h(b_{31} + b_{32})f_i(y^{(k)})$$

$$+ h^2 b_{32}b_{21}\big(\underbrace{\partial_1 f_i \cdot f_1 + \partial_2 f_i \cdot f_2}_{\text{all at } y^{(k)}}\big) + O(h^3)$$

On the other hand

$$y_i(t_k+h) = y_i(t_k) + hy_i'(t_k) + \frac{h^2}{2}y_i''(t_k) + O(h^3)$$

$$= y_i^{(k)} + hf_i(y^{(k)})$$

$$+ \frac{h^2}{2}\big(\underbrace{\partial_1 f_i \cdot \overset{y_1'}{f_1} + \partial_2 f_i \cdot \overset{y_2'}{f_2}}_{\frac{d}{dt}f_i(y(t))\big|_{t_k}}\big) + O(h^3)$$

Conclude for error $= O(h^3)$, need

$$\boxed{\begin{aligned} b_{31} + b_{32} &= 1 \\ b_{32}b_{21} &= \tfrac{1}{2} \end{aligned}}$$

2.

Note that while the k-step linear multistep methods have k eigenvalues when applied
to y' = lambda y, the Runge-Kutta methods have only one eigevalue because they are
single-step methods. So for RK methods the single eigenvalue is easy to obtain -
you just plug lambda y in the RK formula in place of f(y), and when the dust settles you have

y_{k+1} = z y_{k}

where the eigenvalue z is something that will be a polynomial in {h lambda}.
To find the ends of the real part of the region of stability you need to solve

z = 1 or z = -1 (so that |z| = 1) for {h lambda}.

sympy.solve() can solve a polynomial for you.

3. This is sort of a trick question, the way they worded it. Just so you know.