Day -6 (23) Outline 4/20/21

Poisson/Laplace by FD cont'd (5 methods!)

Direct solution

- naive solution — slow
- solution recognizing banded structure

Iterative solution

- Jacobi iteration
- Gauss-Seidel
- SOR

Trying to solve $\boxed{Ax = b}$

Write $A = \underset{\text{strictly lower-tri}}{L} + \underset{\text{diagonal}}{D} + \underset{\text{strictly upper-tri}}{U}$

$$(L + D + U)x = b$$

$$\boxed{Dx = b - (L+U)x}$$

$$\downarrow$$

$$Dx_{k+1} = b - (L+U)x_k \qquad \text{(Jacobi)}$$

A fixed point of this iteration is a solution of $Ax = b$.

$$x_{k+1} = \underbrace{D^{-1}b}_{\equiv c} - \underbrace{D^{-1}(L+U)}_{\equiv T}x \qquad \text{Jacobi}$$

$$(2) \quad \boxed{x_{k+1} = Tx_k + c} \quad \text{an affine iteration}$$

What is important for convergence?

Answer: T

because if x^* is a fixed point

and define $y_k = x_k - x^*$

if you plug this into (2) you will find

$$\boxed{y_{k+1} = Ty_k} \quad \text{linear iteration}$$

Therefore this converges if $\rho(T) = \max_i |\lambda_i| < 1$

ρ: spectral radius

λ_i: eigenvalues of T

Spectrum of T is key.

For our problem

$$\begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array} = \cancel{f_{i,j}}\ 0 \quad \text{Laplace}$$

$$\begin{array}{ccc} & & \\ & -4 & \\ & & \end{array} = \begin{array}{ccc} & -1 & \\ -1 & & -1 \\ & -1 & \end{array}$$

$$\boxed{\begin{array}{ccc} & & \\ & (k+1) & \\ & & \end{array} = \dfrac{\begin{array}{ccc} & (k) & \\ (k) & & (k) \\ & (k) & \end{array}}{4}}$$ Jacobi iteration