

9.3.9

 $u'' + u = f$  by variation of parameters

$$u(0) = 0, \quad u(L) = 0$$

Corresponding homogeneous eqn is

$$u'' + u = 0$$

which has general solution

$$u(x) = a \cos x + b \sin x$$

or, more conveniently

$$u(x) = A \sin(x - \phi)$$

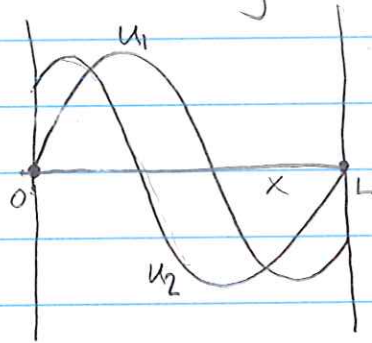
This form is convenient for making it match the BCs.

Let us choose

$$u_1(x) = \sin x, \quad \text{which satisfies } u(0) = 0$$

$$\text{and } u_2(x) = \sin(x-L), \quad \text{which satisfies } u(L) = 0$$

(They look something like this:



Note that the condition

$$L \neq n\pi$$

guarantees  $u_1(L) \neq 0$ ,  
and  $u_2(0) \neq 0$ .

Then we look for a solution of the non-homogeneous equation of the form

$$u(x) = v_1(x) \sin x + v_2(x) \sin(x-L).$$

Then

$$u'(x) = v_1'(x) \sin x + v_1(x) \cos x + v_2'(x) \sin(x-L) + v_2(x) \cos(x-L)$$

Now let's "spend" one of our degrees of freedom

$$\text{to make } \boxed{v_1(x) \sin x + v_2'(x) \sin(x-L) = 0}$$

Then  $u'(x) = v_1(x) \cos x + v_2(x) \cos(x-L)$

and

$$u''(x) = v_1'(x) \cos x + v_1(x) (-\sin x) + v_2'(x) \cos(x-L) + v_2(x) (-\sin(x-L))$$

and

$$\begin{aligned} u''(x) + u(x) &= v_1'(x) \cos x - v_1(x) \sin x + v_2'(x) \cos(x-L) - v_2(x) \sin(x-L) \\ &\quad + v_1(x) \sin x + v_2(x) \sin(x-L) \\ &= v_1'(x) \cos x + v_2'(x) \cos(x-L) \end{aligned}$$

This must be  $f(x)$ , so in all we have

$$v_1'(x) \cos x + v_2'(x) \cos(x-L) = f(x) \quad (*)$$

from previous page →

$$v_1'(x) \sin x + v_2'(x) \sin(x-L) = 0$$

Multiplying the first eqn by  $\sin(x-L)$  and the second eqn by  $\cos(x-L)$ , and subtracting, we eliminate  $v_2'$  and obtain

$$v_1'(x) (\cos x \sin(x-L) - \sin x \cos(x-L)) = f(x) \sin(x-L)$$

$$v_1'(x) = f(x) \sin(x-L) / (\cos x \sin(x-L) - \sin x \cos(x-L))$$

Can we simplify that denominator?  
Thankfully, yes! It is  $-\sin L$ . So

$$v_1'(x) = \frac{-f(x) \sin(x-L)}{\sin L}$$

```
from sympy import *
x,L = symbols('x,L', real=True)
simplify( cos(x)*sin(x-L) - sin(x)*cos(x-L) )
-sin(L)
```

From the second equation (\*), we then have

$$v_2'(x) = \frac{-v_1(x) \sin x}{\sin(x-L)},$$

So

$$v_2'(x) = - \left( \frac{-f(x) \sin(x-L)}{\sin L} \right) \frac{\sin x}{\sin(x-L)}?$$

$$\boxed{v_2'(x) = \frac{f(x) \sin x}{\sin L}}$$

To obtain  $v_1(x)$ ,  $v_2(x)$  we now just need to integrate (using  $x_0$  as the "dummy" integration variable)

$$v_1(x) = \int_0^x \frac{-f(x_0) \sin(x_0-L)}{\sin L} dx_0 + c_1$$

$$v_2(x) = \int_0^x \frac{f(x_0) \sin x_0}{\sin L} dx_0 + c_2$$

To determine the constants  $c_1, c_2$ , we apply the BCs.

$$\bullet \quad u(0) = v_1(0)u_1(0) + v_2(0)u_2(0) = 0$$

$$c_2 \cdot \sin(-L) = 0 \Rightarrow \boxed{c_2 = 0}$$

(again using  $L \neq n\pi$ , so  $\sin L \neq 0$ )

$$\bullet \quad u(L) = v_1(L)u_1(L) + v_2(L)u_2(L) = 0$$

$$\left( \int_0^L \frac{-f(x_0) \sin(x_0-L)}{\sin L} dx_0 + c_1 \right) \sin L = 0$$

Again since we're guaranteed  $\sin L \neq 0$ ,

$$c_1 = \int_0^L \frac{f(x_0) \sin(x_0 - L)}{\sin L} dx_0$$

Thus

$$\begin{aligned} v_1(x) &= \int_0^x \frac{-f(x_0) \sin(x_0 - L)}{\sin L} dx_0 + \int_0^L \frac{f(x_0) \sin(x_0 - L)}{\sin L} dx_0 \\ &= \int_x^L \frac{f(x_0) \sin(x_0 - L)}{\sin L} dx_0 \end{aligned}$$

and the solution of the non-homog. eqn is thus

$$u(x) = \left( \int_x^L \frac{f(x_0) \sin(x_0 - L)}{\sin L} dx_0 \right) \cdot \sin x + \left( \int_0^x \frac{f(x_0) \sin x_0}{\sin L} dx_0 \right) \cdot \sin(x - L)$$

$$\begin{aligned} &= \int_0^x f(x_0) \left[ \frac{\sin x_0 \cdot \sin(x - L)}{\sin L} \right] dx_0 \\ &\quad + \int_x^L f(x_0) \left[ \frac{\sin(x_0 - L) \sin x}{\sin L} \right] dx_0 \end{aligned}$$

Which we can write as  $u(x) = \int_0^L f(x_0) G(x, x_0) dx_0$

if we set

$$G(x, x_0) = \begin{cases} \frac{\sin x_0 \cdot \sin(x - L)}{\sin L} & , x_0 < x \\ \frac{\sin(x_0 - L) \sin x}{\sin L} & , x_0 > x \end{cases}$$

This is the Green's function for the problem.

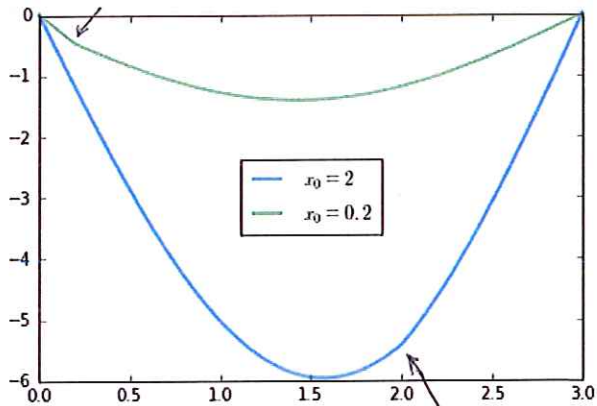
# (x) Plot of Green's function

## 9.3.9 Green's function for ODE BVP

```
%pylab inline
from numpy import *
def G(x,x0):
    Gvals = sin(x0-L)*sin(x)/sin(L)
    xgtx0 = x > x0
    Gvals[xgtx0] = sin(x[xgtx0]-L)*sin(x0)/sin(L)
    return Gvals
```

Populating the interactive namespace from numpy and matplotlib

```
L = 3
x = linspace(0,L,300)
for x0 in [2,.2]:
    plot(x,G(x,x0),label='$x_0=' + str(x0) + '$',lw=2,alpha=0.5,clip_on=False)
legend(loc='center');
```



discontinuity in slope